

Mathematical remarks on transcritical bifurcation in Hamiltonian systems

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Abstract

This article is meant as a mathematical appendix or comment on [1]. We first consider the notion of transcritical bifurcations of fixed points of general area-preserving maps, and then address some questions related to [1] on bifurcation in Poincaré maps of 2-dimensional Hamiltonian systems.

1 Rank-1-bifurcations

Let differentiable (C^∞) functions $Q = Q(q, p, \varepsilon)$ and $P = P(q, p, \varepsilon)$ be defined on an open neighborhood of the origin in (q, p, ε) -space \mathbb{R}^3 . We assume (P, Q) to be symplectic in the (q, p) -coordinates, which means that

$$\det \begin{pmatrix} Q_q(q, p, \varepsilon) & Q_p(q, p, \varepsilon) \\ P_q(q, p, \varepsilon) & P_p(q, p, \varepsilon) \end{pmatrix} \equiv 1, \quad (1)$$

where we have written the partial derivatives as $\frac{\partial Q}{\partial q} =: Q_q$ etc. We then speak of (Q, P) as describing a **symplectic family**. Our bifurcating fixed point shall be the origin of the (q, p) -plane at the value $\varepsilon = 0$ of the parameter, so $Q(0, 0, 0) = 0$ and $P(0, 0, 0) = 0$. At the bifurcation point $(0, 0, 0)$, we assume the Jacobian matrix with respect to the variables (q, p) to have the eigenvalue $+1$. Otherwise, by the implicit function theorem, the fixed point set could be parametrized locally by ε and hence would not bifurcate in the way we wish to study. We will exclude, however, the exceptional case that the Jacobian matrix equals the identity matrix. So by assumption, the eigenspace is a 1-dimensional subspace of the (q, p) -plane. We can always adjust the canonical coordinates, for example by a simple rotation, to have the eigenspace as the q -axis. Then at $(0, 0, 0)$ we have

$$\begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} = \begin{pmatrix} 1 & Q_p \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_p \neq 0 \quad (2)$$

in these coordinates. The total fixed point set

$$F := \{(q, p, \varepsilon) \mid Q(q, p, \varepsilon) = q; P(q, p, \varepsilon) = p\} \quad (3)$$

is the inverse image of the origin $(0, 0)$ in \mathbb{R}^2 under the map $(Q - q, P - p)$, and the Jacobian matrix of this map at $(0, 0, 0)$ is

$$\begin{pmatrix} Q_q - 1 & Q_p & Q_\varepsilon \\ P_q & P_p - 1 & P_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & Q_p & Q_\varepsilon \\ 0 & 0 & P_\varepsilon \end{pmatrix}. \quad (4)$$

If we were to look for generic bifurcations, we could now impose the condition that this matrix be of rank 2, or $P_\varepsilon \neq 0$ at $(0, 0, 0)$. Then F would be locally a smooth

1-dimensional submanifold, tangent to the q -axis at the bifurcation point, and if we require, as a further generic condition, $\varepsilon|F$ to have a nondegenerate extremum at this point, we would arrive at the notion of an *extremal fixed point* or *saddle-node bifurcation* in the sense of K. R. Meyer [2].

Instead, in this note we will be interested in bifurcations with matrix (4) being of rank 1. Among these **rank-1-bifurcations** we will distinguish various types by further conditions. For a convenient coordinate description, let us adjust the canonical (q, p) -coordinates one step further by an ε -dependent translation

$$\begin{aligned}\tilde{q} &= q \\ \tilde{p} &= p - c\varepsilon\end{aligned}\tag{5}$$

where the constant c is defined by $c := -Q_\varepsilon/Q_p$ at $(0, 0, 0)$. Then the \tilde{q} -axis still is the eigenspace at the bifurcation point, and the Jacobian matrix of $(\tilde{Q} - \tilde{q}, \tilde{P} - \tilde{p})$ at $(0, 0, 0)$ is simplified to

$$\begin{pmatrix} \tilde{Q}_{\tilde{q}-1} & \tilde{Q}_{\tilde{p}} & \tilde{Q}_\varepsilon \\ \tilde{P}_{\tilde{q}} & \tilde{P}_{\tilde{p}-1} & \tilde{P}_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & Q_p & 0 \\ 0 & 0 & 0 \end{pmatrix}.\tag{6}$$

So let (q, p) be such coordinates to begin with. To have a name for them, we introduce the following terminology.

Definition 1: Symplectic coordinates (q, p) shall be called **adapted coordinates** for a rank-1-bifurcation at $(0, 0, 0)$, if

$$\begin{pmatrix} Q_{q-1} & Q_p & Q_\varepsilon \\ P_q & P_{p-1} & P_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & Q_p & 0 \\ 0 & 0 & 0 \end{pmatrix}\tag{7}$$

at the bifurcation point, and $Q_p \neq 0$ there. \square

Then the inverse image of 0 under $Q - q$, let's denote it by

$$X := \{(q, p, \varepsilon) \mid Q(q, p, \varepsilon) = q\},\tag{8}$$

will locally at $0 = (0, 0, 0)$ be a smooth surface, with the (q, ε) -plane as tangent plane T_0X at this point. The fixed point set F is contained in X , it is the inverse image

$$F = \{(q, p, \varepsilon) \in X \mid P(q, p, \varepsilon) = p\},\tag{9}$$

of 0 under the restriction $(P - p)|_X$ of $P - p$ to X . As $(0, 0, 0)$ is a critical point of this restriction, we naturally turn to its Hessian quadratic form on the (q, ε) -plane for information about the local behavior of $P - p$ on X . And since $(0, 0, 0)$ is critical not only for the restriction, but also for the function $P - p$ itself, we know that this quadratic form is simply described by the Hessian matrix of $P - p$, hence of P , at $(0, 0, 0)$ with respect to the (q, ε) -coordinates. This is how this matrix enters the following definition.

Definition 2: A rank-1-bifurcation shall be called **regular**, if in adapted coordinates the Hessian

$$\begin{pmatrix} P_{qq} & P_{q\varepsilon} \\ P_{q\varepsilon} & P_{\varepsilon\varepsilon} \end{pmatrix} \quad \text{at} \quad (0, 0, 0)\tag{10}$$

of P with respect to q and ε is nondegenerate. Depending on whether it is definite or indefinite, we speak of the rank-1-bifurcation as being **definite** or **indefinite**. An indefinite rank-1-bifurcation will be called a **cross-bifurcation**, and a cross-bifurcation is called **transcritical**, if the q -axis is not contained in the zero set of the quadratic form, that is if $P_{qq} \neq 0$ at $(0,0,0)$, in adapted coordinates. \square

The definition does not depend on the *choice* of the adapted coordinates. As can be shown, the Hessians from different choices of adapted coordinates are equivalent (up to sign) as quadratic forms.

2 The fixed point set: branches and traces

Let a regular rank-1-bifurcation in adapted coordinates be given. We now denote the restriction $(P - p)|X$ by $\psi : X \rightarrow \mathbb{R}$. So we are interested in $\psi^{-1}(0) \subset X$, because this is the fixed point set F of the bifurcation. The Hessian quadratic form

$$\text{Hess}_\psi : T_0X \rightarrow \mathbb{R} \quad (11)$$

of ψ at the bifurcation point is given by the matrix (10). In coordinates (q, ε) on X , Taylor expansion to second order of ψ at this point gives $\psi \approx \frac{1}{2}\text{Hess}_\psi$ up to higher order terms. But since the Hessian is assumed to be nondegenerate, we can do better than that. By the Morse Lemma, see for instance [3], we can find local coordinates on X , approximating (q, ε) in first order, in which ψ actually *coincides* with $\frac{1}{2}\text{Hess}_\psi$. More precisely, there is a diffeomorphism $f : \Omega \rightarrow \Omega'$ from an open neighborhood Ω of the origin in the tangent plane of X to an open neighborhood Ω' of the bifurcation point in X itself, such that $f(0) = 0$ and

$$\psi(f(v)) = \frac{1}{2}\text{Hess}_\psi(v) \quad (12)$$

for all $v \in \Omega$, and the differential $df_0 : T_0X \rightarrow T_0X$ is the identity. Thus up to this diffeomorphism, the fixed point set of the bifurcation looks locally the same as the zero set $\text{Hess}_\psi^{-1}(0)$ of the Hessian, which is a single point in the definite and a pair of straight lines in the indefinite case. Summing up:

Proposition 1: *In a sufficiently small neighborhood of a regular rank-1-bifurcation point $(0,0,0)$ in the (q, p, ε) -space, the fixed point set consists of a single point if the bifurcation is definite and it is the union $A \cup B$ of two smooth 1-dimensional submanifolds, intersecting at the bifurcation point, if the bifurcation is indefinite that is in the case of a cross-bifurcation. Moreover, if a and b denote the tangents to A and B at the bifurcation point, then $a \neq b$, and the intersection of the plane spanned by a and b in the (q, p, ε) -space with the plane defined by fixing the bifurcation parameter value that is with the (q, p) -plane $\varepsilon = 0$ is the 1-dimensional eigenspace at the bifurcating fixed point. The bifurcation is transcritical, if and only if this eigenspace is different from a and from b .* \square

Recall that in adapted coordinates the eigenspace is the q -axis and the plane spanned by a and b is T_0X , the (q, ε) -plane.

Now let for a moment $(Q(p, q, \varepsilon), P(p, q, \varepsilon))$ denote any local symplectic family with a fixed point at $(0,0,0)$, without assuming it to be rank-1-bifurcating or bifurcating at all. If there is a differentiable map $\alpha : I \rightarrow \mathbb{R}^2$, defined on an open

interval $I \subset \mathbb{R}$ around 0, written in coordinates as $\alpha(\varepsilon) = (q(\varepsilon), p(\varepsilon))$, such that $\alpha(0) = (0, 0)$ and

$$Q(q(\varepsilon), p(\varepsilon), \varepsilon) = q(\varepsilon) \quad \text{and} \quad P(q(\varepsilon), p(\varepsilon), \varepsilon) = p(\varepsilon) \quad (13)$$

for all $\varepsilon \in I$, then we call its graph

$$A := \{(q(\varepsilon), p(\varepsilon), \varepsilon) \mid \varepsilon \in I\} \subset \mathbb{R}^3 \quad (14)$$

a **fixed point branch** of $(0, 0, 0)$, regardless of what other fixed points of the symplectic family might exist. For instance, if a transcritical bifurcation is considered in a suitable small neighborhood, then its fixed point set will consist of two branches A and B as in Proposition 1, but if an indefinite rank-1-bifurcation is *not* transcritical, then only one of its fixed point lines, say A , will be a branch, while B , being tangent to $\varepsilon = 0$, cannot be parametrized differentiably by ε .

If again A is a fixed point branch of $(0, 0, 0)$ in some symplectic family, then the trace of the Jacobian matrix, taken at each point of the branch, defines a differentiable real valued function $\text{Tr}_A : I \rightarrow \mathbb{R}$, so

$$\text{Tr}_A(\varepsilon) := Q_q(q(\varepsilon), p(\varepsilon), \varepsilon) + P_p(q(\varepsilon), p(\varepsilon), \varepsilon). \quad (15)$$

Then the eigenvalue at $(0, 0, 0)$ is 1 if and only if $\text{Tr}_A(0) = 2$. Under what additional condition will $(0, 0, 0)$ be an indefinite rank-1-bifurcation point, i.e. a cross-bifurcation?

Proposition 2: *If a fixed point branch A of $(0, 0, 0)$ in a symplectic family satisfies $\text{Tr}_A(0) = 2$, then $(0, 0, 0)$ will be a cross-bifurcation point if and only if $\text{Tr}'_A(0) \neq 0$.*

PROOF: So let a fixed point branch A of $(0, 0, 0)$ in a symplectic family (Q, P) be given, and $\text{Tr}_A(0) = 2$ be assumed. We have to show (a): If $\text{Tr}'_A(0) \neq 0$, then $(0, 0, 0)$ is a cross-bifurcation point, and conversely (b): If $(0, 0, 0)$ is a cross-bifurcation point, then $\text{Tr}'_A(0) \neq 0$.

Proof of (a): We first choose the symplectic (q, p) -coordinates in such a way, that the q -axis is contained in the eigenspace, and the ε -axis $q = p = 0$ is tangent to the branch at $(0, 0, 0)$. Then we already have

$$\begin{pmatrix} Q_q - 1 & Q_p & Q_\varepsilon \\ P_q & P_p - 1 & P_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & Q_p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (16)$$

at $(0, 0, 0)$, but we do not yet know if $Q_p \neq 0$, and we know nothing about the Hessian of P , except $P_{\varepsilon\varepsilon}(0, 0, 0) = 0$, which follows from differentiating $P(q(\varepsilon), p(\varepsilon), \varepsilon) = p(\varepsilon)$ twice at $\varepsilon = 0$, note $q'(0) = p'(0) = 0$ by our choice of coordinates. So it remains to show that $Q_p \neq 0$ and $P_{q\varepsilon} \neq 0$ at the bifurcation point. Now since (Q, P) is a symplectic family, we have

$$\det \begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} = 1, \quad (17)$$

everywhere, not only at fixed points. So if u denotes any of our three variables q, p or ε , we obtain

$$\det \begin{pmatrix} Q_{qu} & Q_p \\ P_{qu} & P_p \end{pmatrix} + \det \begin{pmatrix} Q_q & Q_{pu} \\ P_q & P_{pu} \end{pmatrix} = 0 \quad (18)$$

everywhere, and thus at the special point $(0, 0, 0)$ we get

$$\det \begin{pmatrix} Q_{qu} & Q_p \\ P_{qu} & 1 \end{pmatrix} + \det \begin{pmatrix} 1 & Q_{pu} \\ 0 & P_{pu} \end{pmatrix} = 0. \quad (19)$$

For later reference, let us call this the

Determinant derivative formula: *If $Q_q = P_p = 1$ and $P_q = 0$ at some point in a symplectic family, then if u is any of the three variables q , p , or ε we have*

$$Q_{qu} + P_{pu} = Q_p P_{qu} \quad (20)$$

at this particular point.

Presently we apply the formula for $u := \varepsilon$. Since $\text{Tr}'_A(0) = Q_{q\varepsilon}(0, 0, 0) + P_{p\varepsilon}(0, 0, 0)$ because of $q'(0) = p'(0) = 0$, we have $\text{Tr}'_A(0) = Q_p P_{q\varepsilon}$ at $(0, 0, 0)$, so $Q_p P_{q\varepsilon} \neq 0$ by our assumption $\text{Tr}'_A(0) \neq 0$, and (a) is proved.

Proof of (b): Now we assume $P_{qq}P_{\varepsilon\varepsilon} - P_{q\varepsilon}^2 < 0$ at $(0, 0, 0)$ in adapted coordinates, and we have to show that this implies $\text{Tr}'_A(0) \neq 0$. The branch A must be tangent to the (q, ε) -plane at the bifurcation point, which means $p'(0) = 0$, and a change to adapted coordinates of the type

$$\begin{aligned} \tilde{q} &= q - c_1 \varepsilon \\ \tilde{p} &= p \end{aligned} \quad (21)$$

with a suitable constant c_1 will even make the ε -axis tangent to A , with the Hessian still being indefinite in the new coordinates. The trace function $\text{Tr}_A(\varepsilon)$ remains unchanged anyway, as it is independent of the coordinate choice. So we may assume $q'(0) = p'(0) = 0$ for the branch from the start. As before, this implies $\text{Tr}'_A(0) = Q_{q\varepsilon} + P_{p\varepsilon}$ and $P_{\varepsilon\varepsilon} = 0$ at $(0, 0, 0)$. Since we have a rank-1-bifurcation, we know $Q_p \neq 0$, and since it is regular, $P_{q\varepsilon} \neq 0$ follows from $P_{\varepsilon\varepsilon} = 0$ at $(0, 0, 0)$. The determinant derivative formula is applicable and shows $\text{Tr}'_A(0) = Q_p P_{q\varepsilon}$ and hence $\text{Tr}'_A(0) \neq 0$, which completes the proof of (b) and of Proposition 2. \square

As we have seen in Proposition 1, locally at a *transcritical* bifurcation point the fixed point set consists of two branches A and B . Since the functions $\text{Tr}_A(\varepsilon) - 2$ and $\text{Tr}_B(\varepsilon) - 2$ both change sign at $\varepsilon = 0$, both branches change their stability properties from elliptic to hyperbolic or vice versa, and in fact in opposite directions. More precisely

Proposition 3: *If A and B are the branches of a transcritical bifurcation point, then $\text{Tr}'_A(0) + \text{Tr}'_B(0) = 0$.*

PROOF: In adapted coordinates we have $p'_A(0) = p'_B(0) = 0$ anyway, since locally the fixed point set is contained in the surface X tangent to $p = 0$. Changing the adapted coordinates by a suitable transformation of the type (21), we can also obtain $q'_A(0) + q'_B(0) = 0$. In these coordinates, we have $P_{q\varepsilon}(0, 0, 0) = 0$. On the other hand, $\text{Tr}'_A(0) + \text{Tr}'_B(0) = Q_{qq}q'_A(0) + Q_{q\varepsilon} + P_{pq}q'_A(0) + P_{p\varepsilon} + Q_{qq}q'_B(0) + Q_{q\varepsilon} + P_{pq}q'_B(0) + P_{p\varepsilon} = 2(Q_{q\varepsilon} + P_{p\varepsilon})$, but this is $2Q_p P_{q\varepsilon}$ by the determinant derivative formula, and the proposition follows. \square

3 Fork-like bifurcations

Let us now have a look at those cross-bifurcations, which are *not* transcritical. Then only one of the two fixed point lines A and B of Proposition 1 will be a *branch* in the technical sense (14), while the other, say B , is tangent to the q -axis. Thus the function $B \rightarrow \mathbb{R}$ given by the restriction $\varepsilon|_B$ of the family parameter to B is critical at the bifurcation point. It may in fact happen that $\varepsilon|_B \equiv 0$, the most degenerate possibility. But here we introduce a terminology for the *least* degenerate case.

Definition 3: Let a cross-bifurcation which is not transcritical be given. Then if the restriction $\varepsilon|_B$ of the bifurcation parameter to one of the local fixed point lines has a nondegenerate extremum at the bifurcation point, we speak of a ***fork-like*** bifurcation. \square

It is the same condition on B as is required of the fixed point set in the definition of an ordinary saddle-node rank-2-bifurcation. But in our case, the additional fixed point branch A is present, while in a saddle-node bifurcation the fixed point set is locally just a single line.

Since B in the non-transcritical case is tangent to the eigenspace, it can be parametrized as $(q, p_B(q), \varepsilon_B(q))$ by q in adapted coordinates, with $p'_B(0) = \varepsilon'_B(0) = 0$. The fork-like condition then just says $\varepsilon''_B(0) \neq 0$. In the surface X of (8), the line B locally looks like a parabola up to higher order terms, tangent to $\varepsilon = 0$.

The name is meant to refer to the well-known *pitchfork bifurcations* in symplectic families. These are bifurcations of fixed points that have eigenvalue -1 and as such do not fall under our heading. But if we consider the *iterated* family instead, $(Q(Q(q, p, \varepsilon), P(q, p, \varepsilon)), P(Q(q, p, \varepsilon), P(q, p, \varepsilon)), \varepsilon)$ in coordinates, then the fixed point gets the eigenvalue $+1$ and the iterated family bifurcates it fork-like. But being iterations, these are very special fork-like bifurcations. For instance does the original family provide a sort of symmetry, which a fork-like bifurcation in general will not have.

The trace along the fixed point line B of a fork-like bifurcation is of course well-defined as a function Tr_B on B itself, but not like Tr_A as a function of ε , since for small $\varepsilon \neq 0$ we find either none or two points on B , and these two may in fact have different traces. But in a natural way, $\text{Tr}'_B(0)$ is still defined and in a fixed relation to $\text{Tr}'_A(0)$, as the next proposition shows.

Proposition 4: *Let A be the fixed point branch of a fork-like bifurcation and B the fixed point line tangent to $\varepsilon = 0$. Then there exists the limit*

$$\lim_{q \rightarrow 0} \frac{\text{Tr}_B(q, p_B(q), \varepsilon_B(q)) - 2}{\varepsilon_B(q)} =: \text{Tr}'_B(0) \quad (22)$$

and it satisfies

$$\text{Tr}'_A(0) + \frac{1}{2}\text{Tr}'_B(0) = 0. \quad (23)$$

PROOF: To determine the limit, we will use second order Taylor expansion of $Q_q(q, p, \varepsilon) + P_p(q, p, \varepsilon)$ in adapted coordinates (7). All partial derivatives now taken

at $(0, 0, 0)$. By the determinant derivative formula we know

$$\begin{aligned} (Q_q + P_p)_q &= Q_p P_{qq} \\ (Q_q + P_p)_p &= Q_p P_{qp} \\ (Q_q + P_p)_\varepsilon &= Q_p P_{q\varepsilon}, \end{aligned} \quad (24)$$

and $P_{qq} = 0$ since the bifurcation is not transcritical. Moreover,

$$\text{Tr}'_A(0) = (Q_q + P_p)_q q'_A(0) + (Q_q + P_p)_p p'_A(0) + (Q_q + P_p)_\varepsilon \quad (25)$$

and $p'_A(0) = 0$, so from (24) and $P_{qq} = 0$ we also get

$$\text{Tr}'_A(0) = Q_p P_{q\varepsilon}. \quad (26)$$

Thus we know the linear terms, and the quotient becomes

$$\begin{aligned} \frac{\text{Tr}_B(q, p_B(q), \varepsilon_B(q)) - 2}{\varepsilon_B(q)} &= Q_p P_{qp} \frac{p_B(q)}{\varepsilon_B(q)} + \text{Tr}'_A(0) \\ &\quad + \frac{1}{\varepsilon_B(q)} \cdot (\text{higher order terms}) \end{aligned} \quad (27)$$

But not many of the higher order terms will contribute to the limit, since

$$\lim_{q \rightarrow 0} \frac{p_B(q)}{\varepsilon_B(q)} = \frac{p''_B(0)}{\varepsilon''_B(0)} \quad \text{and} \quad \lim_{q \rightarrow 0} \frac{q^2}{\varepsilon_B(q)} = \frac{2}{\varepsilon''_B(0)}, \quad (28)$$

and as an intermediate result we obtain that the limit $\text{Tr}'_B(0)$ exists and that

$$\text{Tr}'_B(0) = \text{Tr}'_A(0) + \frac{Q_p P_{qp} p''_B(0) + Q_{qqq} + P_{pqq}}{\varepsilon''_B(0)}. \quad (29)$$

To prove (23), we have to prove that the quotient on the right hand side of this last equation (29) equals $-3\text{Tr}'_A(0)$, and thus by (26) what remains to be shown is

$$3Q_p P_{q\varepsilon} \varepsilon''_B(0) + Q_p P_{qp} p''_B(0) + Q_{qqq} + P_{pqq} = 0. \quad (30)$$

Now we use the fixed point property

$$\begin{aligned} Q(q, p_B(q), \varepsilon_B(q)) &= q \\ P(q, p_B(q), \varepsilon_B(q)) &= p_B(q) \end{aligned} \quad (31)$$

of B . Differentiating the first equation twice and the second three times at $q = 0$, we get

$$Q_{qq} + Q_p p''_B(0) = 0 \quad (32)$$

and

$$P_{qqq} + 3P_{q\varepsilon} \varepsilon''_B(0) + 3P_{qp} p''_B(0) = 0. \quad (33)$$

Inserting this into equation (30), which we have to prove, we find that (30) is equivalent to

$$Q_{qqq} + P_{pqq} = Q_p P_{qqq} - 2Q_{qq} P_{pq} \quad (34)$$

at $(0, 0, 0)$. But this turns out to be a consequence of a second order determinant derivative formula: we start from the symplectic property (17), apply $\partial^2/\partial q^2$, put

in what we know about the first and second order partial derivatives of Q and P at $(0, 0, 0)$, and out comes (34). Thus Proposition 4 is proved. \square

From the calculations of the proof, let us preserve the formula

$$\varepsilon_B''(0) = \frac{3Q_{qq}P_{qp} - Q_pP_{qqq}}{3Q_pP_{q\varepsilon}}, \quad (35)$$

which is a consequence of (32) and (33). Since in its derivation the non-vanishing of $\varepsilon_B''(0)$ was not used, we have as a

Corollary: *A non-transcritical cross-bifurcation is fork-like if and only if*

$$3Q_{qq}P_{qp} \neq Q_pP_{qqq} \quad (36)$$

at $(0, 0, 0)$ in adapted coordinates. \square

4 Fixed point branches given by librating orbits

Now let $H : M \rightarrow \mathbb{R}$ be an autonomous Hamiltonian on a 4-dimensional symplectic manifold and $\gamma : \mathbb{R} \rightarrow M$ a periodic orbit with period $T > 0$ of the Hamiltonian flow. Choose any 3-dimensional submanifold $\Sigma \subset M$ which is being intersected transversally by γ at time $t = 0$. Then on a sufficiently small neighborhood $U \subset \Sigma$ of $\gamma(0)$ in Σ , the Poincaré map $\text{Poinc} : U \rightarrow \Sigma$ is well-defined by following the orbits until they hit Σ again after travelling approximately the period time T of γ . Let $E_0 := H(\gamma(0))$ be the energy of the fixed point $\gamma(0)$ and write $U_E := U \cap H^{-1}(E)$ and $\Sigma_E := \Sigma \cap H^{-1}(E)$. After making Σ smaller if necessary, the individual U_E and Σ_E will be nondegenerate subsurfaces of M , with the Poincaré map defining a symplectic map $U_E \rightarrow \Sigma_E$ for each E . Now we introduce a coordinate ε for the energy, like $\varepsilon = E - E_0$, and extend it to local coordinates (q, p, ε) for Σ such that (q, p) are symplectic coordinates on each Σ_E . Then the Poincaré map will be a symplectic family, described by two functions $Q(q, p, \varepsilon)$ and $P(q, p, \varepsilon)$ in these coordinates. All this is of course well-known, recalled here only to introduce notation.

Often used is the following sort of ‘automatic’ choice of Σ and the coordinates q and p . Let $H = H(x, y, p_x, p_y)$ be given on $M = \mathbb{R}^4$. If the periodic orbit γ satisfies $\dot{y}(0) \neq 0$, then the 3-dimensional subspace $y = y_0 := y(0)$ in \mathbb{R}^4 can be taken for a start to find Σ . Of course, far away from the point $\gamma(0)$ this space may have bad properties with respect to the Hamiltonian flow. But if we choose Σ as a sufficiently small open neighborhood of $\gamma(0)$ in this 3-space, then not only the Poincaré map and the energy surfaces Σ_E will be defined as described, but also the projection of each Σ_E to the (x, p_x) -plane will be symplectic and injective, which means that x and p_x can be used as the symplectic coordinates q and p on Σ_E .

The transcritical bifurcations studied in [1] are related to straight-line librating orbits of Hamiltonians of the form

$$H(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y). \quad (37)$$

Let γ be such a straight-line periodic orbit, without loss of generality projecting to the y -axis, oscillating there between two values $y_1 < y_2$, with $x(0) = 0$ and

$y_1 < y(0) < y_2$ and an energy E_0 . As just recalled, we use

$$\begin{aligned} q &:= x \\ p &:= p_x \\ \varepsilon &:= E - E_0 \end{aligned} \tag{38}$$

as coordinates to describe the ‘automatic’ Poincaré map by a symplectic family $(Q(q, p, \varepsilon), P(q, p, \varepsilon))$, with the fixed point $(0, 0, 0)$ corresponding to the orbit γ . What do we know about this symplectic family?

Straight-line librating orbits in Hamiltonian systems of type (37) always come in one-parameter families. The given orbit γ satisfies

$$\dot{p}_x(t) = -\frac{\partial V}{\partial x}(0, y(t)) \equiv 0, \tag{39}$$

hence at least we know $\frac{\partial V}{\partial x}(0, y) = 0$ for all y with $y_1 \leq y \leq y_2$. If V is polynomial or real-analytic, this implies

$$\frac{\partial V}{\partial x}(0, y) = 0 \quad \text{for all } y \in \mathbb{R}. \tag{40}$$

In this case the whole (y, p_y) -plane $x = p_x = 0$ in \mathbb{R}^4 is invariant under the Hamiltonian flow, and the flow lines are just those of the 1-dimensional Hamiltonian system on the (y, p_y) -plane given by

$$h(y, p_y) = \frac{1}{2}p_y^2 + V(0, y). \tag{41}$$

By standard regularity arguments, the orbit γ must be embedded in a family of neighboring closed orbits of this 1-dimensional system, one for each energy in an interval (E_1, E_2) with $E_1 < E_0 < E_2$. Thus we have a family of librating orbits of the original 2-dimensional system on the y -axis over¹ (E_1, E_2) . In our Poincaré symplectic family it corresponds to a fixed point branch

$$A := \{(0, 0, \varepsilon) \mid \varepsilon_1 < \varepsilon < \varepsilon_2\}. \tag{42}$$

For a branch A , the trace function $\text{Tr}_A : (\varepsilon_1, \varepsilon_2) \rightarrow \mathbb{R}$ is defined, and if it happens to be that $\text{Tr}_A(0) = 2$ and $\text{Tr}'_A(0) \neq 0$, then by Proposition 2 we have a cross-bifurcation point, which may or may not be transcritical. Examples of both cases have been presented numerically in [1].

Hamiltonians of type (37) have time-reversal symmetry, and in [1] the destruction of a transcritical bifurcation by a symmetry breaking perturbation is described. What role does time-reversal symmetry play in transcritical bifurcations? Could we *preserve* a transcritical bifurcation under a symmetry breaking perturbation, and

¹Here we assumed (40). What if $\frac{\partial V}{\partial x}(0, y)$ may be non-zero for $y > y_2$ or $y < y_1$? Then the neighboring closed orbits on the *inside* of γ , those with energies $E_1 < E < E_0$, still are librating orbits of the original system as well, and define a fixed point branch in the Poincaré family with $q = p = 0$ and $\varepsilon_1 < \varepsilon < 0$. The orbits of the 1-dimensional system which are on the outside of γ , with energies $E_0 < E < E_2$, need not be orbits of the 2-dimensional system. But then we simply shift attention from E_0 to an energy E'_0 between E_1 and E_0 . So in any case, we may assume a fixed point branch A as in (42) in the Poincaré symplectic family be given, describing a family of neighboring straight-line librating orbits on the y -axis.

conversely, can a transcritical bifurcation be *destroyed* by a symmetric perturbation? The answer to the first question is yes, as we are going to show now.

5 Transcritical bifurcation in unsymmetric Hamiltonian systems

Again we start from a Hamiltonian $H(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$ with a straight-line librating orbit γ , embedded in a family of such orbits, all projecting to the y -axis and thus constituting a fixed point branch $A = 0 \times 0 \times (\varepsilon_1, \varepsilon_2)$ in the Poincaré symplectic family as described above, with $(0, 0, 0)$ representing γ . For convenience and with little loss of generality we assume (40), that was $\frac{\partial V}{\partial x}(0, y) = 0$ for *all* $y \in \mathbb{R}$. But our main assumption now will be that $(0, 0, 0)$ is a cross-bifurcation point in the Poincaré family. We then speak of γ as of a **cross-libration** or of a **transcritical libration**, if this cross-bifurcation happens to be transcritical.

Now let the Hamiltonian depend on an additional small parameter δ . Such a function $\overline{H}(x, y, p_x, p_y, \delta)$ will be called a **perturbation** of H or of the transcritical libration, if $\overline{H}(x, y, p_x, p_y, 0) = H(x, y, p_x, p_y)$. Then we have a two-parameter Poincaré symplectic family

$$(\overline{Q}, \overline{P}) = (\overline{Q}(q, p, \varepsilon, \delta), \overline{P}(q, p, \varepsilon, \delta)), \quad (43)$$

defined on an open neighborhood of $(0, 0, 0, 0)$ in the $(q, p, \varepsilon, \delta)$ -space \mathbb{R}^4 in the usual way, with q and p from x and p_x .

Definition 4: We say that a perturbation \overline{H} of a cross-libration is **cross-preserving**, if there are differentiable functions $q(\delta)$, $p(\delta)$ and $\varepsilon(\delta)$, defined on an interval $(-\delta_0, \delta_0)$, with $q(0) = p(0) = \varepsilon(0) = 0$, such that for any fixed $\delta \in (-\delta_0, \delta_0)$ the point $(q(\delta), p(\delta), \varepsilon(\delta))$ is a cross-bifurcation point in the symplectic ε -family given by $(\overline{Q}, \overline{P})$ at the fixed δ . \square

Note that a transcritical libration will stay transcritical under a cross-preserving deformation, for small enough δ_0 .

There is a simple strategy to find cross-preserving perturbations. All we have to do is to make sure that for all sufficiently small δ the (y, p_y) -plane $0 \times \mathbb{R} \times 0 \times \mathbb{R}$ is invariant under the Hamiltonian flow on \mathbb{R}^4 that is defined by \overline{H} for fixed δ . In other words, if

$$\frac{\partial \overline{H}}{\partial p_x}(0, y, 0, p_y, \delta) = 0 \quad \text{and} \quad \frac{\partial \overline{H}}{\partial x}(0, y, 0, p_y, \delta) = 0 \quad (44)$$

for sufficiently small δ and all (y, p_y) , then \overline{H} will preserve the transcriticality. Why? Because then the closed orbits neighboring γ will define a 2-parameter fixed point branch \overline{A} of $(\overline{Q}, \overline{P})$ given by functions $q(\varepsilon, \delta)$ and $p(\varepsilon, \delta)$, defined on a neighborhood of $(0, 0)$ in the (ε, δ) -plane. We know

$$\text{Tr}_{\overline{A}}(0, 0) = 2 \quad \text{and} \quad \frac{\partial \text{Tr}_{\overline{A}}}{\partial \varepsilon}(0, 0) \neq 0 \quad (45)$$

by assumption, thus by the implicit function theorem, we get a differentiable function $\varepsilon(\delta)$ with $\varepsilon(0) = 0$ and

$$\text{Tr}_{\overline{A}}(\varepsilon(\delta), \delta) = 2 \quad \text{and} \quad \frac{\partial \text{Tr}_{\overline{A}}}{\partial \varepsilon}(\varepsilon(\delta), \delta) \neq 0 \quad (46)$$

for sufficiently small δ . By Proposition 2, each $(q(\varepsilon(\delta), \delta), p(\varepsilon(\delta), \delta), \varepsilon(\delta))$ is a cross-bifurcation point in the symplectic family of the corresponding fixed δ (and if it is transcritical for $\delta = 0$, it must remain transcritical for sufficiently small δ). So the strategy is sound, and we use it to derive the following proposition.

Proposition 5: *Let $H(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$ be a Hamiltonian with a cross-libration on the y -axis and $\partial V/\partial x(0, y) = 0$ for all $y \in \mathbb{R}$. Let $F(x, y, p_x, p_y)$ be an arbitrary differentiable function with*

$$\frac{\partial F}{\partial p_x}(0, y, 0, p_y) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x}(0, y, 0, p_y) = 0 \quad (47)$$

Then

$$\overline{H}(x, y, p_x, p_y, \delta) := H(x, y, p_x, p_y) + \delta F(x, y, p_x, p_y) \quad (48)$$

is a cross-preserving perturbation, since it obviously satisfies (44). \square

It is a matter of definition, which Hamiltonians should be counted as ‘unsymmetric’ in this context. Making H unsymmetric by a bump far away from the transcritical libration would not illuminate the relation between symmetry and transcriticality. But Proposition 5 shows transcritical bifurcation to occur in the Hamiltonian system $H(x, y, p_x, p_y) + \delta F(x, y, p_x, p_y)$ on \mathbb{R}^4 for fixed sufficiently small $\delta \neq 0$. Since the only condition on $F(x, y, p_x, p_y)$ is the vanishing of the partial derivatives by x and p_x along the (y, p_y) -plane, we may fairly say yes, transcritical bifurcation can occur in *unsymmetric* autonomous 2-dimensional Hamiltonian systems.

6 Destruction of cross-bifurcations

Once more we start with a cross- or a transcritical libration γ on the y -axis of a Hamiltonian system $H(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$ with $\frac{\partial V}{\partial x}(0, y) = 0$ for all $y \in \mathbb{R}$. This time we ask by which perturbations of H the cross-property can be destroyed. By destruction we mean more than just non-preservation:

Definition 5: We say that a perturbation \overline{H} of a cross-libration is **cross-destroying**, if there is a neighborhood of $(0, 0, 0, 0)$ in $(q, p, \varepsilon, \delta)$ -space, in which $(0, 0, 0, 0)$ is the only cross ε -bifurcation point of $(\overline{Q}, \overline{P})$. \square

We will first prove a general destruction criterion for cross-bifurcation in symplectic families.

Proposition 6: *Let $(Q(p, q, \varepsilon), P(p, q, \varepsilon))$ be a symplectic family with a cross-bifurcation at $(0, 0, 0)$. Extend it to a symplectic family $(\overline{Q}(p, q, \varepsilon, \delta), \overline{P}(p, q, \varepsilon, \delta))$ depending on a second parameter δ , with $\overline{Q}(p, q, \varepsilon, 0) = Q(p, q, \varepsilon)$ and $\overline{P}(p, q, \varepsilon, 0) = P(p, q, \varepsilon)$. Then if the vector $(\overline{Q}_\delta(0, 0, 0, 0), \overline{P}_\delta(0, 0, 0, 0))$ does not belong to the eigenspace of (Q, P) at the bifurcation point, there is a neighborhood of $(0, 0, 0, 0)$ in the $(q, p, \varepsilon, \delta)$ -space, in which it is the only ε -cross-bifurcation point.*

PROOF: The assumption simply means that at $(0, 0, 0, 0)$ the Jacobian matrix

$$\begin{pmatrix} Q_q - 1 & Q_p & Q_\varepsilon & \overline{Q}_\delta \\ P_q & P_p - 1 & P_\varepsilon & \overline{P}_\delta \end{pmatrix} \quad (49)$$

of $(\overline{Q} - q, \overline{P} - p)$ is of rank 2, as can be seen most easily in adapted coordinates, where the condition reduces to $\overline{P}_\delta \neq 0$. So the fixed point set \overline{F} , locally at $(0, 0, 0, 0)$, is a smooth surface, and since it contains the two fixed point lines A and B of the given cross-bifurcation, its tangent plane at the point must be spanned by a and b and thus be the (q, ε) -plane in adapted coordinates. In particular, the restriction $\delta|_{\overline{F}}$ is singular at $(0, 0, 0, 0)$.

To prove the proposition, it would be sufficient to show that this is an *isolated* singularity of $\delta|_{\overline{F}}$. Because then at any other fixed point $(q_0, p_0, \varepsilon_0, \delta_0)$ in a neighborhood, the set of fixed points with $\delta = \delta_0$ will be locally a smooth 1-dimensional submanifold of \overline{F} and hence the point can't be an ε -cross-bifurcation point.

To find out, we calculate the matrix of the Hessian quadratic form of $\delta|_{\overline{F}}$ at the singularity. Straightforward calculation using Lagrangian multipliers gives

$$\text{Hess}_{\delta|_{\overline{F}}}(0, 0, 0, 0) = -\frac{1}{\overline{P}_\delta} \begin{pmatrix} P_{qq} & P_{q\varepsilon} \\ P_{q\varepsilon} & P_{\varepsilon\varepsilon} \end{pmatrix} \quad (50)$$

for this matrix, which is now seen to be nondegenerate by assumption, hence the singularity is isolated and Proposition 6 is proved. \square

Let γ be a cross-libration on the y -axis of a Hamiltonian system

$$H(x, y, p_x, p_y) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V_0(x, y) \quad (51)$$

with $\frac{\partial V_0}{\partial x}(0, y) = 0$ for all $y \in \mathbb{R}$. For an attempt to apply Proposition 6 to perturbations

$$\overline{H}(x, y, p_x, p_y, \delta) := H(x, y, p_x, p_y) + \delta F(x, y, p_x, p_y) \quad (52)$$

with an arbitrary perturbation term $F(x, y, p_x, p_y)$, we will have to understand how $(\overline{Q}_\delta(0, 0, 0, 0), \overline{P}_\delta(0, 0, 0, 0))$ does depend on $F(x, y, p_x, p_y)$.

Proposition 7: *Let $T > 0$ denote the period of the cross-libration γ and $y(t)$ its y -component. Define the function $f(y)$ by*

$$f(y) := \frac{\partial^2 V_0}{\partial x^2}(0, y). \quad (53)$$

and let $g_1(y, p_y)$ and $g_2(y, p_y)$ be given by

$$\begin{aligned} g_1(y, p_y) &:= \frac{\partial F}{\partial p_x}(0, y, 0, p_y), \\ g_2(y, p_y) &:= -\frac{\partial F}{\partial x}(0, y, 0, p_y). \end{aligned} \quad (54)$$

Let $(\xi(t), \eta(t))$ be the solution of

$$\begin{aligned} \dot{\xi} - \eta &= g_1(y(t), \dot{y}(t)) \\ \dot{\eta} + f(y(t))\xi &= g_2(y(t), \dot{y}(t)) \end{aligned} \quad (55)$$

with the initial condition $\xi(0) = \eta(0) = 0$. Then \overline{H} satisfies the destruction criterion of Proposition 6 if and only if $(\xi(T), \eta(T))$ is not contained in the eigenspace of the Poincaré map of the undisturbed system at the bifurcation point. \square

What would it take to apply Proposition 7 numerically? First of all, one has to know the component $y(t)$ and its derivative $\dot{y}(t) = p_y(t)$ of the cross-librating orbit γ . Secondly, we will need the fundamental system $(\varphi(t), \psi(t))$ of the homogeneous equation

$$\ddot{\xi} + f(y(t))\xi = 0 \quad (56)$$

with initial conditions $\varphi(0) = \dot{\psi}(0) = 1$ and $\dot{\varphi}(0) = \psi(0) = 0$, and finally we would have to calculate the two numbers

$$\begin{aligned} c_1(T) &:= \int_0^T (\dot{\psi}(t)g_1(y(t), \dot{y}(t)) - \psi(t)g_2(y(t), \dot{y}(t))) dt \\ c_2(T) &:= - \int_0^T (\dot{\varphi}(t)g_1(y(t), \dot{y}(t)) - \varphi(t)g_2(y(t), \dot{y}(t))) dt. \end{aligned} \quad (57)$$

This then is all we need, because as can be shown, the Jacobian matrix of the Poincaré map at the bifurcation point turns out to be

$$\begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} = \begin{pmatrix} \varphi(T) & \psi(T) \\ \dot{\varphi}(T) & \dot{\psi}(T) \end{pmatrix}, \quad (58)$$

and standard calculation of $(\xi(T), \eta(T))$ leads to the following

Corollary: *The perturbation term $F(x, y, p_x, p_y)$ satisfies the destruction criterion of Proposition 6 if and only if*

$$\begin{pmatrix} \varphi(T) - 1 & \psi(T) \\ \dot{\varphi}(T) & \dot{\psi}(T) - 1 \end{pmatrix} \begin{pmatrix} c_1(T) \\ c_2(T) \end{pmatrix} \neq 0. \quad (59)$$

□

Application to $F(x, y, p_x, p_y) = xp_y - yp_x$ leads to the condition

$$\begin{pmatrix} \varphi(T) - 1 & \psi(T) \\ \dot{\varphi}(T) & \dot{\psi}(T) - 1 \end{pmatrix} \begin{pmatrix} \int_0^T y(t)\dot{\psi}(t) dt \\ - \int_0^T y(t)\dot{\varphi}(t) dt \end{pmatrix} \neq 0. \quad (60)$$

For a more systematic approach to the calculation of partial derivatives of the Poincaré map in connection with straight-line librations, see [4].

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